DIFFERENTIAL GEOMETRY II MIDTERM EXAMINATION

Total marks: 40
Attempt all questions
Time: 3 hours (2 pm - 5 pm)

(1) Let G be a topological group which acts continuously on the right on a topological space S (that is, the map $S \times G \to S$ is continuous). Let S/G be the set of orbits of G with the quotient topology. Prove that the natural map $\pi : S \to S/G$ is an open map. If S is Hausdorff (respectively, second countable) then is S/G also Hausdorff (resp, second countable)? Prove or give a counter example. (3+3+2 = 8 marks)

(2) Let M, N be two smooth manifolds. Prove that the tangent space $T_{(m,n)}(M \times N)$ of the product manifold $M \times N$ at a point $(m, n) \in M \times N$ is isomorphic to $T_m M \times T_n N$ (use the projection maps $M \times N \to M$ and $M \times N \to N$). Let G be a Lie group with multiplication map $m : G \times G \to G$. Prove that the differential at the identity $e \in G$ of the multiplication map m is addition, that is $m_{*,(e,e)} : T_e G \times T_e G \to T_e G$ takes (X_e, Y_e) to $X_e + Y_e$ for any $X_e, Y_e \in T_e G$. (4+4 = 8 marks)

(3) State the regular value (or regular level set) theorem, show an application. Deduce the Implicit Function Theorem from this theorem. Let N be a compact manifold, then show that a one-one immersion $f : N \to M$ is an embedding. (4+4+4 = 12 marks)

(4) Define the notion of a smooth vector field on a smooth manifold, give an example. Define the notion of an integral curve to a smooth vector field X on a smooth manifold M starting at a point $p \in M$, give an example. Define the local flow generated by X on M about the point $p \in M$), give an example. Find the maximal integral curve $c(t)$ starting at the point $(a, b) \in \mathbb{R}^2$ of the vector field $X(x, y) = \frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. (3+3+3+3 = 12 marks)